Stability of the toroidal magnetic field in stellar radiation zones

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ABSTRACT

Understanding the stability of the magnetic field in radiation zones is of crucial importance for various processes in stellar interior like mixing, circulation and angular momentum transport. The stability properties of a star containing a prominent toroidal field in a radiation zone is investigated by means of a linear stability analysis in the Boussinesq approximation taking into account the effect of thermal conductivity. The growth rate of the instability is explicitly calculated and the effects of stable stratification and heat transport are discussed in detail. It is argued that the stabilizing influence of gravity can never entirely suppress the instability caused by electric currents in radiation zones although the stable stratification can significantly decrease the growth rate of instability

Subject headings: MHD - instabilities - stars: interiors - stars: magnetic fields - Sun: magnetic fields

1. Introduction

Magnetic fields localized in stellar radiation zones can play an important role in many essential phenomena like mixing, angular momentum transport and formation of the solar tachocline, for instance.

It is rather uncertain which field can be present in the radiation zone. Observations provide upper limits on the allowed strength of the magnetic field in the radiation zone of the Sun. For instance, helioseismological measurements suggest an upper limit $B < 4 \times 10^7$ G as an order of magnitude estimate (see, e.g., Friedland & Gruzinov 2004). The recently measured oblateness of the Sun (Codier & Rozelot 2000) implies that the strength of the magnetic field is lower than 7×10^6 G. The observations of the splitting of solar oscillation frequencies provide the upper limit of $B < 3 \times 10^5$ G for a toroidal field near the base of the convective zone (Antia et al. 2000). For other stars estimates are less certain and only theoretical upper limits can be derived.

The possible origin of the field (if it exists) is also unclear. Likely, a hydromagnetic dynamo cannot operate in the radiation zone, where no strong flows are available to sustain a vigorous dynamo action. Perhaps, relic magnetic fields acquired by the star at the early stage of evolution can persist there. These type of fields could have formed, for instance because of differential rotation, which could have stretched a weak diffuse primordial seed field (see, e.g., Dicke 1979) into a dominant toroidal field. Due to a high conductivity, the ohmic decay is very slow and the decay time can exceed several billion years. Therefore, once formed, the large-scale relic field would survive in the radiation zone during the life-time of a star.

The magnetic field, however, can evolve in a radiation zone not only due to ohmic dissipation but also because of the development of various instabilities. For instance, in differentially rotating radiation zones, the magnetorotational instability can occur. Most likely, however, the magnetized radiative zones rotate rigidly and the stability of magnetic configurations is determined by various current-driven instabilities. Such instabilities are well studied in cylindrical geometry in the context of laboratory fusion research (see, e.g., Freidberg 1973, Goedboed 1971, Goedbloed & Hagebeuk 1972). For example, the stability properties of a pure toroidal field B_φ are determined by the parameter $\alpha = d \ln B_\varphi(s)/d \ln s$ where s is the cylindrical radius. The field is unstable to axisymmetric perturbations if $\alpha > 1$ and to non-axisymmetric perturbations if $\alpha > -1/2$ (Tayler 1973a,b, 1980). Note, however, that the addition of a even relatively weak poloidal field alters substantially the stability properties of the magnetic configuration. For example, if the poloidal field is uniform and relatively weak, the instability condition reads $\alpha > -1$ at variance with the condition of instability for a purely toroidal field (see, e.g., Knobloch 1992; Dubrulle & Knobloch 1993) which predicts that an unstable toroidal field configuration has $\alpha > -1/2$. In astrophysical conditions, the instability caused by electric currents might have various characteristic properties (see Bonanno & Urpin 2008a,b). In the presence of both azimuthal and axial fields, non-axisymmetric disturbances with large azimuthal wavenumbers m turn out to be most rapidly growing. Unstable disturbances exhibit a resonant character, i.e. the wave vector

$\vec{k} = (m/s)\vec{e}_\varphi + k_z\vec{e}_z$ approximately satisfies the condition of magnetic resonance, $\vec{B} \cdot \vec{k} = 0$ where k_z is the wavevector in the axial direction and $\vec{B}$ is the magnetic field. The length scale of this instability depends on the ratio of poloidal and azimuthal field components and it can be very short, while the width of the resonance turns out to be very narrow. For this reason, its excitation in direct numerical simulations can be problematic.

Stability of the spherical magnetic configurations is studied in much less detail and even the overall stability properties of radiation zones are rather unclear. Braithwaite & Nordlund (2006) have studied the stability of a random initial field in the stellar radiative zone by means of direct numerical simulations and found that the stable magnetic configurations generally have the form of tori with comparable poloidal and toroidal field strengths. Numerical modeling by Braithwaite (2006) confirmed that the toroidal field with $B_\varphi \propto s$ or $\propto s^2$ is unstable to the $m = 1$ mode as it was predicted by Tayler (1973). However, even a purely toroidal field can be stable in the region where it decreases rapidly with s. Note that a purely toroidal field cannot be stable throughout the whole star because the stability condition for axisymmetric modes ($\alpha < 1$) is incompatible with the condition that the electric current in the z-direction has no singularity at $s \to 0$ which implies $\alpha > 1$. The stability of the toroidal field in rotating stars has been considered by Kitchatinov (2008) and Kitchatinov & Rüdiger (2008) who argued that the magnetic instability is essentially three-dimensional and determined the threshold field strength at which the instability sets. Estimating this threshold in the solar radiation zone, the authors impose the upper limit on the magnetic field ≈ 600 G.

In this paper, we consider the stability of the toroidal field in radiation zones by taking into account stratification and thermal conductivity. It turns out that magnetic configurations can be stable or unstable depending on the radial profile of the toroidal field. We argue that stable stratification can substantially decrease the growth rate of the instability but cannot suppress it entirely.

2. Basic equations

Consider the stability of an axisymmetric toroidal magnetic field in the radiation zone using a high conductivity limit. We work in spherical coordinates (r, θ, φ) with the unit vectors ($\vec{e}_r$, $\vec{e}_\theta$, $\vec{e}_\varphi$). We assume that the toroidal field depends on r and θ, $B_\varphi = B_\varphi(r, \theta)$. In the incompressible limit, the MHD equations read

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} + \vec{g} + \frac{1}{4\pi\rho}(\nabla \times \vec{B}) \times \vec{B}, \tag{1}$$

$$\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{v} \times \vec{B}) = 0, \tag{2}$$

$$\nabla \cdot \vec{v} = 0, \quad \nabla \cdot \vec{B} = 0, \tag{3}$$

where $\vec{g}$ is gravity. In the basic state, the gas is assumed to be in hydrostatic equilibrium, then

$$\frac{\nabla p}{\rho} = \vec{g} + \frac{1}{4\pi\rho}(\nabla \times \vec{B}) \times \vec{B}. \tag{4}$$

We assume that the magnetic energy is subthermal and therefore $\vec{g}$ is approximately radial. The equation of thermal balance reads in the Boussinesq approximation

$$\frac{\partial T}{\partial t} + \vec{v} \cdot (\nabla T - \nabla_{ad} T) = \nabla \cdot (\kappa \nabla T), \tag{5}$$

where κ is the thermal diffusivity and $\nabla_{ad} T$ is the adiabatic temperature gradient.

We consider a linear stability. Small perturbations will be indicated by subscript 1, while unperturbed quantities will have no subscript. Linearizing Eqs.(1)-(3) and (5), we take into account that small perturbations of the density and temperature in the Boussinesq approximation are related by $\rho_1/\rho = -\beta(T_1/T)$ where β is the thermal expansion coefficient. For small perturbations, we use a local approximation in the θ-direction and assume that their dependence on θ is proportional to $\exp(-il\theta)$, where $l \gg 1$ is the longitudinal wavenumber. Since the basic state is stationary and axisymmetric, the dependence of perturbations on t and φ can be taken in the exponential form as well. Then, perturbations are proportional to $\exp(\sigma t - il\theta - im\varphi)$ where m is the azimuthal wavenumber. The corresponding wavevectors are $k_\theta = l/r$ and $k_\varphi = m/r \sin\theta$, respectively. The dependence on r should be determined from Eqs.(1)-(3), (5). For the sake of simplicity, we assume that unperturbed ρ and T are approximately homogeneous in the radiation zone. This assumption does not change the main conclusions qualitatively but substantially simplifies calculations. We also neglect the effect of rotation. As we know from the work of Pitts and Tayler (1986), rotation itself cannot remove instabilities of the interchange type, although it can affect the shape the unstable modes. For this reasons we argue that the conclusions of our investigation cannot be altered by the inclusion of rotation.

Eliminating all variables in favor of v_{1r} and T_1, we obtain the following set of two coupled equations

$$(\sigma^2 + \omega_A^2) v_{1r}'' + \left(\frac{4}{r}\sigma^2 + \frac{2}{H}\omega_A^2 \right) v_{1r}' + \left[\frac{2}{r^2}\sigma^2 - k_\perp^2(\sigma^2 + \omega_A^2) \right. \tag{6}$$

$$\left. + \frac{2}{r}\omega_A^2 \left(\frac{1}{H}\frac{k_\perp^2}{k_\varphi^2} - \frac{2}{r}\frac{k_\theta^2}{k_\varphi^2}\frac{\sigma^2}{\sigma^2 + \omega_A^2} \right) \right] v_{1r} = -k_\perp^2 \beta g \sigma \frac{T_1}{T},$$

$$\frac{\kappa}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{T_1}{T} \right) \right] - (\sigma + \kappa k_\perp^2)\frac{T_1}{T} = \frac{\omega_{BV}^2}{\beta g} v_{1r}, \tag{7}$$

where the prime denotes a derivative with respect to r and

$$\omega_A^2 = \frac{k_\varphi^2 B_\varphi^2}{4\pi\rho}, \quad \omega_{BV}^2 = -\frac{g\beta}{T}(\nabla_{ad}T - \nabla T)_r, \tag{8}$$
$$k_\perp^2 = k_\theta^2 + k_\varphi^2, \quad \frac{1}{H} = \frac{\partial}{\partial r}(rB_\varphi).$$

Some general stability properties can be derived directly from Eqs.(6)-(7). Consider perturbations with a very short radial wavelength for which one can use a local approximation in the radial direction, such as $v_{1r} \propto \exp(-ik_r r)$, where k_r is the radial wavevector. If $k_r \gg \max(k_\theta, k_\varphi)$, then Eqs.(6)-(7) can be reduced with the accuracy in terms of the lowest order in $(k_r r)^{-1}$ to the following set

$$-(\sigma + \kappa k^2)\frac{T_1}{T} = \frac{\omega_{BV}^2}{\beta g} v_{1r}, \qquad k_r^2(\sigma^2 + \omega_A^2)v_{1r} = k_\perp^2 \beta g \sigma \frac{T_1}{T}, \tag{9}$$

where $k^2 = k_r^2 + k_\perp^2$. The corresponding dispersion relation reads

$$\sigma^3 + \kappa k^2 \sigma^2 + \left(\omega_A^2 + \frac{k_\perp^2}{k^2}\omega_{BV}^2\right)\sigma + \kappa k^2 \omega_A^2 = 0. \tag{10}$$

The conditions that at least one of the roots has a positive real part (unstable mode) is determined by the Routh criterion (see, e.g., Aleksandrov, Kolmogorov, & Laurentiev 1985). For a particular case of a cubic equation, these conditions can be easily obtained, for example, from expressions derived by Urpin & Rüdiger (2005). Since the quantities κ and ω_A^2 are positively defined, the only non-trivial condition of instability is $\omega_{BV}^2 < 0$ that is not satisfied in the radiation zone by definition. Therefore, modes a with short radial wavelength are always stable to the current-driven instability contrary to the conclusion obtained by Kichatinov (2008) and Kichatinov & Rüdiger (2008).

3. Numerical results

We assume that the radiation zone is located at $R_i \geq r \geq R$. The toroidal field can be represented as

$$B_\varphi = B_0 \psi(x) \sin\theta, \tag{11}$$

where $x = r/R$ and B_0 is the characteristic field strength; $\psi \sim 1$ is a function of the radius alone. The dependence of ψ on x is uncertain in in the radiation zone and, in this work, we consider only the case where ψ increases with x. Other possibilities will be considered in a forthcoming paper. We parametrize $\psi(x)$ with the following dependence

$$\psi(x) = \left(D_1 \exp\frac{x-1}{d} + D_2\right) \tag{12}$$

where d characterizes the width of a field distribution and D_1 and D_2 are chosen so that $\psi(x = 1) = 1$ and $\psi(x = x_i) = 0$ as shown in Fig.1; $x_i = R_i/R$. We choose x_i to be equal to 0.01 from computational reasons. We have verified that our results are basically insensitive to the precise value of x_i as long as it is close to the center. The situation where the field reaches its maximum at the outer boundary can model, for example, the radiative interior of a star with a convective envelope. In this case, the bottom of the convection zone is the location of the toroidal field generated by a dynamo action which can penetrate into the radiation zone. This situation can also mimic the toroidal field in the liquid core of neutron stars. Likely, the magnetic field of these objects is generated by turbulent dynamo during the very early phase of the evolution when the neutron star is subject to hydrodynamical instabilities (see, e.g., Bonanno et al. 2003). Dynamo induced by turbulent motions generates the magnetic field of complex topology including small scale fields (Urpin & Gil 2004). Large-scale dynamo is most efficient in the surface layers where the density gradient is maximal. Therefore, the generated field increases outward and reaches its maximum in the outer layers (Bonanno et al. 2005, 2006). This magnetic field can be subject to current-driven instabilities after the end of the unstable phase.

Introducing the dimensionless quantities

$$\omega_{A0}^2 = \frac{B_0^2}{4\pi\rho R_2^2}, \quad \Gamma = \frac{\sigma}{\omega_{A0}}, \quad \delta^2 = \frac{\omega_{BV}^2}{\omega_{A0}^2}, \quad \varepsilon = \frac{\omega_T}{\omega_{A0}}, \tag{13}$$

where $\omega_T = \kappa/R_2^2$, we can transform Eqs.(6)-(7) into a dimensionless form. These equations with the corresponding boundary conditions describe the stability problem as a non-linear eigenvalue problem. Fortunately, the main qualitative features of this problem are not sensitive to the choice of boundary conditions. That is why we choose the simplest conditions and assume that $v_{1r} = T_1 = 0$ at $r = R_i$ and $r = R$. Note that the parameter δ is large in radiation zones but, most likely, ε is relatively small if the magnetic field is not very weak. In calculations, we suppose δ and ε to be constant through the radiation zone.

To illustrate the stabilizing effect of stratification on the instability, we plot in Fig.2 the growth rate as a function of the stratification parameter δ for several eigenmodes with a different number of nodes in the radial direction. In these calculations, we neglect the thermal conductivity, therefore, the stabilizing effect of gravity is most pronounced. The growth rate is always maximal for the fundamental eigenmode ($n = 0$) but rapidly decreases with an increasing number of nodes n. Note that such a rapid decrease of Γ with the number of an eigenmode is typical also for the case of neutral stratification $\delta = 0$ (or no gravity $g = 0$). The conclusion that Γ decreases rapidly with n confirms our result obtained from Eqs.(6)-(7) in a short wavelength approximation (see Section 2) that modes with a short radial lengthscale (large n) should be stable. This conclusion is at variance with the statement by

Kitchatinov & Rüdiger (2008) that the most rapidly growing modes correspond to $n \sim 10^3$. According to our calculations, the fundamental mode turns out to be most rapidly growing. However, even the instability of this mode is entirely suppressed if $\delta \gtrsim 9$. The effect of a thermal conductivity can change the properties of a current-driven instability qualitatively. In Fig.3, we plot the dependence of the growth rate on the parameter stratification δ^2 for three different values of the thermal conductivity, corresponding to $\varepsilon = 10^{-2}$, 2×10^{-2}, and 4×10^{-2}. The behaviour of all curves is qualitatively similar: the growth rate is ≈ 1 at small δ decreases as

$$\Gamma \propto \delta^{-2} \tag{14}$$

for $\delta^2 > 100$ or, in dimensional form,

$$\sigma \propto \omega_{A0} (\omega_{A0}/\omega_{BV})^2. \tag{15}$$

This dependence can be obtained also directly from the basic equations. The stabilizing effect of gravity in a linear theory is determined by the first term on the right hand side of Eq.(1). Linearization of this term yields $(\nabla p/\rho)_1 \approx -\vec{g}\rho_1/\rho \approx \vec{g}T_1/T$ since $p_1 \approx 0$ in the Boussinesq approximation. Perturbations of the temperature are determined by thermal balance Eq.(7). Near the threshold of the instability when σ is small, we have $T_1/T \approx -(\omega_{BV}^2/\beta g \kappa k_\perp^2) v_{1r}$. Therefore, the stabilizing contribution of gravity in Eq.(1) is of the order of $(\omega_{BV}^2/\kappa k_\perp^2) v_{1r} \sim \delta^2 (\omega_{A0}^2/\kappa k_\perp^2) v_{1r}$. On the other hand, the destabilizing effect of electric currents in Eq.(1) is given by the Lorentz force. The order of magnitude estimate of the Lorentz force yields $\sim B_\varphi B_{1\varphi}/4\pi\rho r$. Since the toroidal field is inhomogeneous in the basic state, perturbations of the toroidal field are produced basically by perturbations of the radial velocity and $B_{1\varphi} \sim (B_\varphi/\sigma H) v_{1r}$ where H is the radial lengthscale of the magnetic field. Using this expression, we obtain that the Lorentz force is of the order of $\sim (\omega_{A0}^2/a\sigma) v_{1r}$ where $a = H/R$. Equating the stabilizing contribution of gravity to the destabilizing contribution of the electric current, we obtain

$$\sigma \sim \omega_{A0} \, \frac{\varepsilon}{a\delta^2} \, (k_\perp^2 R_2^2). \tag{16}$$

If ε is fixed, this expression yields the dependence $\Gamma \propto \delta^{-2}$ that follows from numerical calculations.

The relation (14) implies that even a very strong stable stratification cannot entirely suppress the instability. The growth rate turns out to be non-vanishing even for large δ at variance with the case where thermal conductivity is neglected (see Fig.2).

4. Conclusions

We have considered the effects of stratification and thermal conductivity on the stability of a toroidal field in stellar radiation zones. The magnetic configuration with a predominantly toroidal field can be formed, for example, due to differential rotation during the early stage of stellar evolution if the star has got even a weak seed field.

It turns out that stable stratification can suppress the current induced instability of the toroidal field if perturbations are not influenced by the thermal conductivity (very small ε). The instability does not arise if the Brunt-Väisälä frequency is greater than $\sim 9\omega_{A0}$. Since ω_{BV} is typically high in radiative zones ($\sim 10^{-3} - 10^{-4}$ s^{-1}) the instability sets in only if the field is very strong ($\geq 10^6 - 10^7$ G). Higher eigenmodes are suppressed stronger than the fundamental one and perturbations with short radial wavelength are always stable.

The thermal conductivity drastically changes the character of the instability. This concerns particularly the behavior near the threshold of instability where the growth rate is small. It turns out that the growth rate is non-vanishing for any stratification. Even very strong gravity cannot suppress the instability entirely but it only decreases the growth rate. This sort of behavior is typical for perturbations with any wavevector $k_\perp$ and can be easily understood. A destabilizing effect of electric currents in the momentum equation (1) originates from the last term on the r.h.s. and is proportional to a perturbation of the magnetic field, B_1. A perturbation B_1 is related to v_1 by $B_1 \propto v_1/\sigma$ as it follows from the linearized induction equation. A stabilizing influence of gravity in the linearized momentum equation is proportional to a perturbation of the temperature, T_1. If the growth rate is small, then we can obtain from thermal balance equation (5) that $T_1 \propto v_1/\omega_T$. Comparing the destabilizing and stabilizing effects, we see that stable stratification can never suppress the instability at variance with the $\kappa = 0$ case.

A decrease of σ caused by stratification is inversely proportional to the Brunt-Väisälä frequency. If gravity is strong but the magnetic field is weak, the instability develops very slowly. Generally, for a sufficiently weak magnetic field, the growth rate can be comparable to the inverse life-time of a star. It should be noted also that, most likely, the field does not decay to zero because of this instability. When the field becomes weaker, the growth rate of the instability decreases and the field cannot decay to values smaller than those resulting from a growth rate of the order of the inverse life-time of a star. Therefore, a weak field can generally be only slowly changed during the life of the star, although its radial profile can be unstable.

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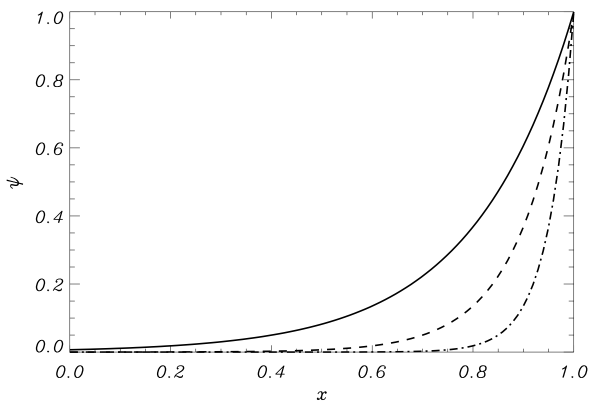

Fig. 1.— The dependence of the toroidal field on the spherical radius for $d = 0.2$ (solid line), 0.1 (dashed). and 0.05 (dot-dashed).

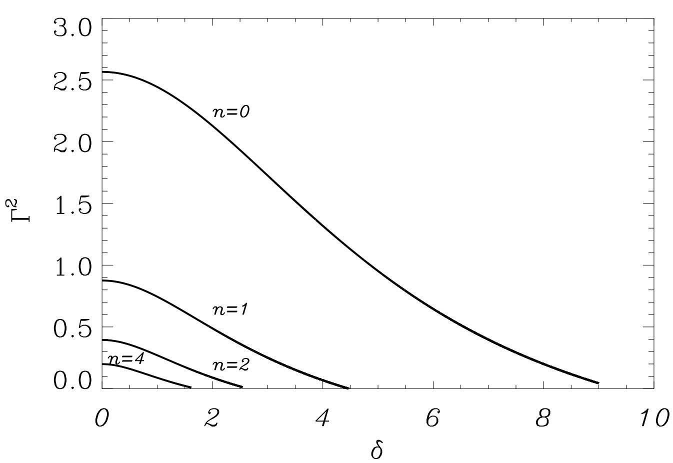

Fig. 2.— The dimensionless growth rate as a function of δ for several eigenmodes with various number of nodes n in the radial direction, for $m = 1$, $l = 20$, $d = 0.1$ and $\varepsilon = 0$.

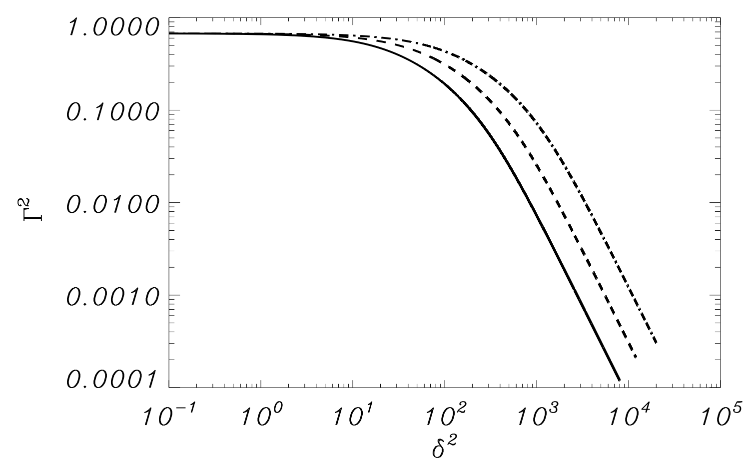

Fig. 3.— The dimensionless growth rate as a function of δ^2 for the fundamental eigenmode with $m = 1$, $l = 20$, $d = 0.1$, and three values of ε: 10^{-2} (solid line), 2×10^{-2} (dashed), and 4×10^{-2} (dot-dashed).